# EXAMPLES, GENERALISA TION AND PROOF

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*The interplay between generalisations and particular instances - examples - is an essential feature of mathematics teaching and learning. In this paper, we bring together our experiences of personal and classroom mathematics activity, and demonstrate that examples do not always fulfill their intended purpose (to point to generalisations). A distinction is drawn between 'empirical' and 'structural' generalisation, and the role of generic examples is discussed as a means of supporting the second of these qualities of generalisation.*

## IN[TRO]DUCTION

For all learners of mathematics there is the possibility of acquiring new knowledge by reflection on appropriate and relevant experience (and arguably there is no other way). Generalisation - unifying and information-extending insight - is central to such a means of coming-to-know, and may be viewed as a form of inductive reasoning. For the great mathematicians, as well as for novices, mathematics characteristically comes into being by inductive intuition, not by deduction.

> Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted to it for his theorems of the binomial and the principle of universal gravity. (Laplace, 1902, p. 176)
>
> I must admit that I am not in a position to give it a rigorous demonstration [ ... J The examples I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula. (Euler, translated by Polya, 1954, pp. 93-95)
>
> The purpose of rigour is to legitimate the conquests of the intuition. (Hadamard, quoted by Burn, 1982, p. 1)

The products of induction are plausible 'truth-estimates' (Rescher, 1980, p. 9), and such conjectures may well be held with conviction. But whereas initial regularity is so often a reliable guide to generality in mathematics, it is not invariably so. Consider the (false) propositions that $n^2+n+4l$ is prime for all $n$ (true for $n=l$ to 39), and that the number of regions ofa circle formed by joining each of $n$ points (irregularly spaced) on the boundary to every other is a power of2 (true for $n=l$ to 5). In these two examples, the mere accumulation of confirming instances misleads. We shall argue that the quality of such empirical evidence is weak for mathematical generalisation, and indicate the need for other sources of conv~ion.

## CONVICTION AND SCEPTICISM

Stamp (undated) recalls teaching a lesson on right-angled triangles. In the first two examples considered - (6,8, 10) and (5, 12, 13) - it was observed by pupils that the area and perimeter had the same numerical value. This led to the conjecture that "this happens every time". Stamp reports that he "denied" that this can be so, and in fact proceeds in the note to deductive demonstration that, with the exception of the given examples, the proposition is universally false.

What do *we* make of this? Tim's reaction to the conjecture was that ifthere were such a connection between the perimeter and area of integer-sided right-angled triangles, then he would already know about it! Therefore (he might reason) there can be no such connection. Whilst this 'mature' mode of reasoning *can* be a reliable guide to induction, it can also be a negative and dangerous reason for scepticism about the remarkable-but-unfamiliar. For example, I [TR] recall vividly that my formal education in mathematics omitted the insight that every quadrilateral tessellates in the plane. The property is so improbable (to me) that I would have expected to know its truth, yet I did not.

Induction is essential for mathematics, but it is not sufficient; the "conquests ofthe intuition" are potentially fallible.

Liz reports two incidents with lower sixth A level mathematics students.

> During a reporting-back session following an exploration of the absolute value function, Lome makes the assertion that the graph of $y$ = If(x)1 is the same as the graph of $y$ = *f(lxl)* for every function **f** I am unsure whether he is right and I try to think of counter-examples. I suggest that he plots $y$ = 12x + 11 and $y$ = 21xl + 1.

Lome had considered five or six examples of functions in coming to this inductive conclusion. I was doubtful, but not because the number of examples considered was too small. Like Tim, I felt that such a striking result would already have been known to me! I also have an image of the graphs of modulus functions that involves points with undefined gradient, where the graph is 'reflected back on itself. My initial, vague feeling of unease formed itself into a counter-example. Then I could see that Lome's examples may have been sufficient in number but of 'the wrong kind'.

Whatever my intuition was which made me doubt the truth of Lome's statement, there was no such doubt in Lome's mind. I can account for this difference in two ways. First, he had less experience of the modulus function on which to draw. Secondly, he was less cautious of inductive reasoning. His schooling had often put him in the position of needing to trust conclusions from inductive reasoning in mathematics without considering the strength of other reasons for conviction.

A few days later ſ recorded what I saw as a similar incident from a class lesson:

> I am talking to the whole class about the way in which they derived the equation of a circle with radius 2 ~ centre (3,5). I have written the equations ~(x - 3i + (} _5)2 = 2 and (x - 3)2 + (y - 5)2 = 4 on the board. I ask 'where did the 3 the 4 and the 5 come from in this (the second) equation?' Trevor replies that the 4 is the diameter of the circle.

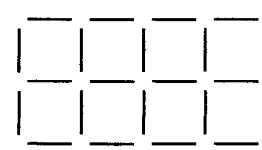
ſ had intended to draw the students' attention to the *structure* of the derivation of this particular equation, with the eventual aim that they would appreciate the form $(x - a)2 + (y - b)2 = r^2$ for the equation of a circle. I was expecting them to base an answer to my question on their recall of the *procedure* by which they had derived the equation. But Trevor seems to be making an *empirical* generalisation from one case, rather than recalling the derivation procedure as I had hoped. I see his statement as a generalisation because he says that 4 is 'the diameter of the circle' and not simply twice 2. His answer relies on seeing that 4 is twice the radius, rather than seeing that 4 is the radius squared, and that it results from the squaring 0Eration which was part of the process of obtaining the equation. He focused his attention on *num ical patterns* rather than *structural relationships.* By contrast, I knew that the constant term in this e uation could not, in general, be equal to the diameter of the circle, since I knew it to be eq al to the square of the radius.

EMPIRICAL AND STRUCTURAL GENERALISA TION

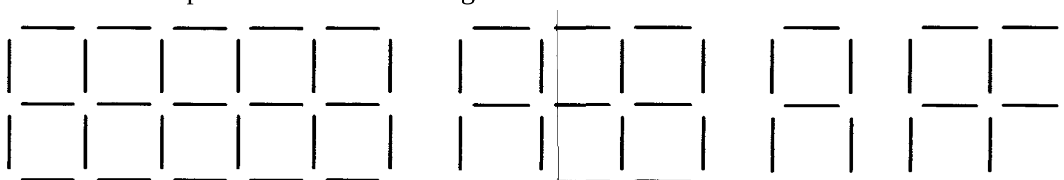
In both of the cases above, the student makes a (sometimes tentative) inductive generalisation. In each case, Liz was sceptical about their conclu ions on the grounds of other sources of conviction. In the second, Trevor generalised from the part· cular circle equation in a way that was different from that intended. We use the terms 'empiric l' and 'structural' to describe, respectively, the form of generalisation Lome and Trevor made and t at which Liz had expected. In using these terms, we emphasise that one form of generalisation is ac ieved by considering the form of results, whist the other is made by looking at the underlying mea ings, structures or procedures.

This distinction is illuminated by Liz's notes o~ the following problem:

> The picture (right) shows a rectangle made up of two rows of four columns of squares outlined by matches. How many matches would be needed to make a rectangle with *R* rows and *C* columns?



When I first worked on this problem, I decided ~o simplifY by holding the number of rows constant. I held *R* as 2 and produced a series of diagrams!such as these:



From my diagrams I produced this table of results:

| No. of columns (*C*) | No. of matches (*M*) |
|---|---|
| 4 | 22 |
| 5 | 27 |
| 3 | 17 |
| 1 | 7 |
| 2 | 12 |

I saw that the results in my table fitted the rule $M = 5C + 2$. My trust in this formula for all positive whole number values of C was based first on the results in my table. Secondly, I was confident in it because it was of the form I was expecting. By this I mean that I expected a relationship to exist between $M$ and C , and previous experience led me to expect the relationship to be linear.

Next I changed the value of $R$ to 3 and, with the aid of one or two diagrams, convinced myselfthat $M$ and C now fitted the rule $M = 7C + 3$. Similarly, I found that, for $R = 4$, $M = 9C + 4$ and, for $R = 5$, $M = llC + 5$. For these, I needed fewer diagrams and tabulated results, because my conviction about these formulae from sources other than my table of results was greater each time. Having established that linear relationships held for $R = 2$ and $R = 3$, I needed only two results in the case $R = 4$ in order to be convinced that I had the correct formula.

Now a pattern was emerging that suggested that a general rule was $M = (2R + I)C + R.$

Again, in moving from these separate formulae for different values of $R$ to one which incorporated variations in $R$, I based my conviction first on the four formulae I had identified in the special cases $R =$ 2,3,4,5. But I had also the anticipation that such a general formula would exist, would be linear in $R$ and in C and would be symmetrical with respect to the two variables.

Finally, seeing beyond the particular numerical features of a diagram so as to perceive it as a 'generic' representative ofthe general (see below), I was able to see that I could count the number of vertical and horizontal matches as follows:

> there are $C + 1$ columns of vertical matches, each containing $R$ matches; there are $R + 1$ rows of horizontal matches, each containing C matches; therefore there are altogether $(C + I)R + (R + I)C$ matches.

This line of argument confirmed the rule which I had arrived at empirically. The argument is *structural,* because it is based on a way of counting the matches in this configuration. By contrast, my first line of argument is *empirical* because it is based (predominantly) on a pattern in the table of results: it argues 'for small values of Rand C the number of matches is given by $(C + I)R + (R + I)C$ so it seems reasonable that this will be the case for all positive whole number values'. Inductive reasoning of this first kind can be a useful way of *producing* conjectures, but in the absence of other sources of conviction it may point to erroneous conclusions.

## GENERIC EXAMPLES

Sometimes, structural generalisation can be achieved by a type of proof by generic example. This mode is discussed in the literature (Mason and Pimijl, 1984; Balacheff, 1988), but little attention has been given to the pedagogy or the epistemic effectiveness of such an approach.

> The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number. (Mason and Pimm, 1984, p. 284)

For example, a generic proof that $\sum_{n=1}^{2k} n = k(2k+1)$ - for all positive integers $k$ - might be effected by reference to the case k=10, showing how the sum can be evaluated as ten pairs (1 and 20, 2 and 19 and so on), each with sum 21. Despite the generic pedagogic *intention* behind such an argument, it may not necessarily be received by the student with the intended generality. Trevor's perception of the equation of the circle with radius 2 and centre (3,5) is a case in point.

Our evidence in this respect is mixed: some undergraduate students in the first term of a Mathematics/Education course were introduced to the well-known 'Stairs' investigation. This problem concerns the number of ways of ascending a flight of $n$ stairs in combinations of ones and twos. The Fibonacci sequence readily emerges in the data, and these students were asked to consider why this is the case - in effect, whether the obvious inductive inference is as valid as it is convincing. One student, Kim, gave an account of why it is that the number of ways for 6 stairs will be equal to the sum of the number of ways for the previous two numbers of stairs. To investigate whether Kim's explanation was perceived as particular or generic, the students were asked to complete the questionnaire below (with spaces for students to write their responses) ..

CLIMBING STAIRS IN ONES AND TWOS

Observation: The number of ways for 6 stairs **[13]** is equal to the sum of the number of ways for 5 stairs [**8**] and the number of ways for 4 stairs [**5**].

Explanation: This is because, in ascending 6 stairs, the first step must be a one or a two. If it is a one, there are 5 stairs left, and there are **8** ways of climbing these 5 stairs. If it is a two, there are 4 stairs left, and there are **5** ways of climbing these 4 stairs. Therefore there are **8+5** ways of ascending 6 stairs.

1. Are you happy with the above explanation i.e. is it convincing?

2. Does the above explanation help to convince you that the number of ways for 15 stairs will be equal to [the number of ways for 14 stairs] + [the number of ways for 13 stairs]?

   If you answered YES, why does the explanation for 6 stairs convince you for 15 stairs?

   If you answered NO, what would you need, in order to be convinced?

3. Does the first explanation [for 6 stairs] convince you that the number of ways *for any number* of stairs will be equal to the sum of the number of ways for the previous two numbers of stairs?

   If you answered YES, why does the explanation for 6 stairs convince you for any number of stairs? If you answered NO, what would you need, in order to be convinced?

Of 17 students, the (anonymous) responses of 15 indicated that the particular example - the explanation for 6 stairs - was, for them, generic in relation to other numbers of stairs. The following responses are typical:

[Student A] If you start with a one, you have 14 left. If you start with a two, you have 13 left. So the sum of these two will form the same formula as for 6 stairs.

[Student B] We could re-write the explanation in terms of *n, n-1* and *n-2* where *n=6* stairs in the explanation and *5=n-1* and *4=n-2.* So we see we had the correct method in the explanation above.

Of the two who were unconvinced (questions 2 and 3), one was explicit about lack of conviction about the initial particular explanation (for 6 stairs), being slightly unsure that it accounted for all possibilities. The other seemed to require further confirming instances of the generalisation before s/he could "accept it".

## SUMMARY

We recognise that inductive reasoning of a quasi-empirical character motivates students, and lends an authentic air of discovery to the mathematics classroom. Such activity may transcend empirical speculation if explanation is available to the student as a structural generalisation of some kind. A generic example which successfully "speaks the generality" (Mason and Pimm, 1984, p. 284) for the audience has the quality of such a structural generalisation.

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