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LINKING THE VISUAL AND THE SYMBOLIC: A MICROGENETIC ANALYSIS OF STUDENTS' EVOLVING APPROACHES TO GENERALISING PROBLEMS

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Abstract

In this paper, we describe our attempts to adopt a methodology consonant with a theory of learning based on constructivism and 'making of connections'. The methodology is being developed as part of the research project Visualisation, Computers and Learning[1]. *In this project we have been investigating student approaches to a sequence of algebraic problems presented with visual information. We are documenting the trajectory of visual and symbolic approaches, attempting to identify the form in which they occur, why they occur and if they are inter-connected. The sequence of activities includes work on the computer and we are exploring if and how interactions with the software connect with other approaches. To illustrate our methodology, we present data from two students who worked through the problems in different mathematical settings.*

Background

Despite studies suggesting that visual approaches to mathematics have considerable potential for supporting meaningful learning (see for example Tall, 1985; 1987; Tall and Vinner, 1981; Vinner and Dreyfus, 1989; Dreyfus, 1990; Bishop, 1989), visualisation is, in general, seen only as a transitory step on the way to real mathematics. The mathematical agenda is identified with symbolic representation to the virtual exclusion of the visual mode of mathematical expression. However, computers offer the potential to operate on geometric images with the kind of rigour which has hitherto been reserved for the symbolic. Graphic computer screen representations of mathematical objects and relationships can be acted upon directly and the ensuing changes in the represented relationships observed. Moreover, the situation can be inverted: it is also possible to investigate which actions will lead to a given change in the existing relationships. The result of such actions can be dynamically implemented; actions can be repeated at liberty, with or without changing the parameters of the action and conclusions inferred on the basis of the feedback given by the program. With this new technology will visualisation become a more acceptable practice of mathematics?

In the ESRC funded project *Visualisation, Computers and Learning* we set out to explore the claim that computers can facilitate visualisation in mathematics in ways which offer the prospect of increasing pupil understanding. Specifically our aims are:

- to map students' visualisation strategies in two mathematical domains: geometry and algebra;
- to identify if and how links are made between symbolisation and visualisation;
- to identify if and how both strategies and linkages are influenced by computer use.

In this paper we describe the algebra strand of the project which centred around work on number patterns and sequences. This curriculum area was chosen as it offers the potential for students to interact with visual representations of numeric sequences: the idea being that students

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would make sense of spatial arrangements in ways which enabled them to construct meaningful algebraic expressions. As we wanted to study the influence of different software environments, three parallel algebra task sequences were designed, each corresponding closely to the specifications laid out for levels 3-6 of the National Curriculum (1991).

Task sequences: One task sequence involved paper and pencil activities with the computer added only at the end of the task sequence (as the computer is not mentioned until level 6). This task sequence is therefore described as Computer-Added Task Sequence (CATS). The other two task sequences were designed to integrate computer use throughout the sequence (CITS). The software environments which were involved were spreadsheets and a specifically written Logo Microworld, Mathsticks. When using spreadsheets the students need to communicate with the software through the spreadsheet language which resembles, to some extent, the language of algebra and allows the construction of general relationships. They receive numeric feedback in response to the symbolic expressions but no visual feedback. In contrast in the Mathsticks microworld, in addition to obtaining visual feedback in response to expressions in the symbolic language of Logo, students can also communicate through interaction with icons and receive corresponding symbolic feedback. By using variables students have the choice of specifying general relationships for which they can receive visual and/or numeric feedback.

Each task sequence (CITS and CATS) consisted of five different mathematical settings and case studies were constructed for three groups of four students each working through one of the task sequences. Figure 1 illustrates students' paths through the complete task sequence.

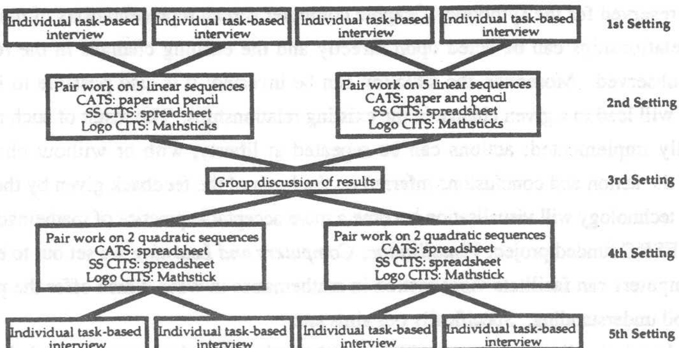

Figure 1: Algebra Task Sequence

In each setting students worked on identifying and expressing general patterns underlying different number sequences presented through both visual and numeric data[2] (see for example, figure 2) and within each problem, the aim was for students to construct a general method to calculate the values for the n^{th} term.

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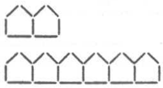
2 Pilot interviews suggest that presenting non-consequetic sequence terms results in greater attention to the visual data.



When there are 2 houses, there are 9 matches

When there are 6 houses, there are 25 matches

Figure 2: 'Houses' sequence

Developing a methodology

We wanted to develop a methodology consonant with a theory of learning that emphasises constructivism and the making of connections. Taking as our starting point students who are relative novices, our long term aim is to examine how all the factors in the learning setting interact to facilitate, or inhibit, students from evolving disconnected fragments of mathematical knowledge into more coherent knowledge systems. With this background in mind, we began by looking at the conceptions and strategies that students apply to these kinds of problems which we identified both from a review of the relevant research literature (e.g. Stacey, 1989; MacGregor and Stacey, 1992; Orton and Orton, 1994) and from the extensive pilot interviews we conducted before undertaking the case studies. We noticed that the research into generalising problems had concentrated on numerical transformation of data and natural language descriptions and/or algebraic representation of this transformation, little explicit attention was focused on visual reasoning – despite the fact that geometric patterns were often the sources of the number sequences under investigation and data presented indicated that at least some of the students were constructing relationships based on visual interpretations (Stacey, 1989). We have chosen to classify approaches as iconic when the students interacted mainly with the visual data and symbolic where this was not the case. In table 1 below we present a summary of the student approaches we identified.

For mathematics educators, the idea behind the number pattern problems is that through expressing generalities in both natural language and symbols, learners will learn to use appropriate symbolic notations in contexts which have meaning for them (see Mason, Burton and Stacey, 1982). The intended outcome is an algebraic expression of a functional relationship which students can understand through reference to the diagrams. An 'expert map' might therefore be expected to involve link the operating on unknown term approach and the intra-term approach. Often, where students use approaches outside the expert map they are seen as less sophisticated mathematically or working at a lower level. It would be easy to read approaches in table 1 as 'levels': The first two, eidetic and counting are not likely to involve algebraic expression, and as you move down the table the kinds of general relationships described become increasingly likely to translate into algebra. We would suggest, however, that there are fundamental flaws in such a reading: not least the assumption that students appropriate the intended learning aims. Data from our pilot interviews suggested that for students, the aim was to describe any general method regardless of whether this was easily expressed algebraically. Students who construct a general method based on, say, a combined terms approach may find it very difficult to represent in an algebraic function form, even if the relationship may be just as efficient for calculating any *given* term in the sequence.

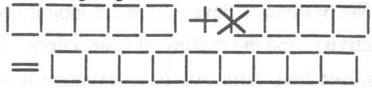
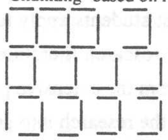
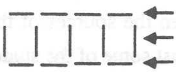
Symbolic	Iconic
Counting:	**Eidetic:**
Count the number of matches in an unstructured way	Focusing on perceptual rather than mathematical properties of the data *"The star is like a cross from noughts and crosses and a religious cross."*
Operating on terms:	**Combined term:**
Calculations using a known term or terms to obtain a target term *1) "there are 16 matches in 5 so there will be 48 in 15, you times by 3"* *2) "to work out 7 I did 10 add 13 because 3 had 10 matches and 4 had*	Combining diagrams of known terms to obtain others 
Operating on differences between terms:	**Inter-term:**
Calculations based on the numerical difference between consecutive terms *1) "4 is 13 because you add 3 each time"* *2) "I added 30 because the difference between 5 and 15 is 10 so you add 10 3's"*	'Chunking' based on relationship between terms 
Operating on unknown term:	**Intra-term:**
Calculations based on a relationship between dependent and independent variables *"You times the number of boxes by 3 and add 1"*	'Chunking' based on relationship within terms 

Table 1: Classification of Student Approaches

It is true that some approaches tend, under certain circumstances, to be associated with erroneous responses. For example, students working on linear sequences of the form $f(n) = an + b$ $(b \neq 0)$ often make false assumptions of direct proportionality between terms, while students who focus on differences sometimes assume that the n^{th} term can be calculated without taking n_0 into account (e.g. Stacey 1989). However, both these approaches are correct for other sequence patterns and it is likely that students' early experiences involve these very sequences. It is also quite possible that, particularly for the latter approach, erroneous responses might reflect more on students' difficulties in translating a correct iterative method into an algebraic expression.

Maps of Student Approaches: We wanted to develop a means through which we could represent the way in which student approaches evolved over the course of the task sequence. Rather than seeing this as reflective of progression through a series of stages, we decided to construct student maps which would allow us to consider all the approaches students use and whether any explicit connections through which students justify the products of one approach by references to another are made. To enable us to trace the path of each individual from first interview to last interview we constructed one map of student approaches for each different setting. Would their approaches remain consistent throughout, or would they vary and if so what factors might have influenced this: task, software tools, and/or interaction with others?

A Snapshot of our Results

To illustrate the ongoing process of data analysis, in figures 3 and 4 we present the first and last interview maps for Lesley (a spreadsheet CITS student) and Amber (a Mathsticks CITS student).

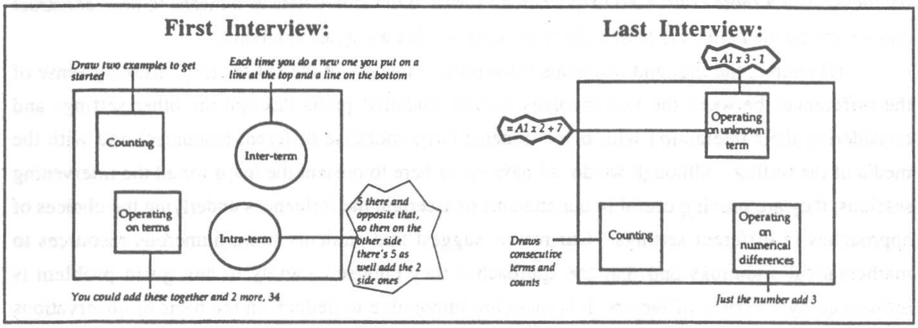


Figure 3: Lesley's First and Last Interview Maps

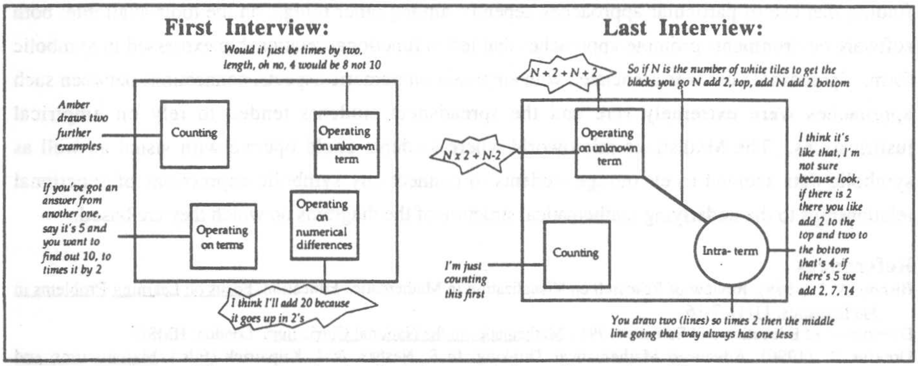


Figure 4: Amber's First and Last Interview Maps

Key to maps: ○ iconic approach □ symbolic approach ✧ student's final general method
— connection — link to evidence for approach or connection

The maps show that in their first interviews both students used a number of different approaches and made no connections between these. Both students were able to construct appropriate generalisations which enabled them to calculate the number of matches provided they were given the required term. Neither represented their generalisations algebraically although Lesley's approach more easily lends itself to this. Lesley made use of some iconic strategies whereas Amber focused primarily on numeric data.

Examination of their final interview MAPS suggest that both students approached the activities rather differently by the end of the task sequence. Both students now chose an expression based on operating with unknown terms for their generalisations and both adopted symbolic language to communicate this generalisation: Lesley employed the spreadsheet code even though she no longer had access to the software, while Amber constructed algebraic expressions. However, while Lesley actually made more use of iconic data at the beginning of the task sequence than at the end, Amber had developed a more connected map whereby she used the iconic data as a means to both construct and justify the mathematical relationships conveyed by her algebraic symbolism.

Of course, the first and last maps show only snapshots of the entire story. Making sense of the differences between the two involves tracing students' paths through the other settings and considering their interaction with other students (who mobilise different resources) and with the media of the setting. Although we do not have space here to present the maps for all the intervening sessions, they are proving useful in our attempts to ascertain the influences underlying the choices of approaches in different settings. Our results suggest that students bring numerous resources to mathematical situations and that the approaches they choose to apply to any given problem is influenced by a variety of factors. It is therefore impossible to deduce on the basis of observations from one setting that a student is incapable of using any particular approach. On the contrary, we are finding that use of particular approaches depends, among other things, on the tools available: both software environments promote approaches that led to functional relationship expressed in symbolic form. Spreadsheet use was associated with emphasis on numeric aspects, connections between such approaches were extremely rare and the spreadsheet students tended to rely on empirical justifications. The Mathsticks microworld where students could operate with visual as well as symbolic data seemed to encourage students to connect any symbolic expressions of functional relationships to the underlying mathematical structure of the diagrams on which they are based.

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