INDUCTIVE REASONING

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This is a theoretical paper, in the quasi-empiricist tradition. It considers mathematical generalisation in the well-trodden context of inductive reasoning. Much of the vast literature on inductive reasoning belongs to the philosophy of science; by contrast, mathematics is supposed to be essentially deductive. Such a view flies in the face of mathematical heuristic. Questions about how we come to 'know' things in mathematics, and why we believe things for which our evidence is at best partial, are usefully examined within an inductive framework.

INTRODUCTION

Mathematical generalisation is a particular form of the epistemic phenomenon of *induction*, and is properly and usefully considered from that broader perspective. The term 'induction' is derived from the Latin rendering (using *ducere*, to lead) of Aristotle's *epagoge* (*epi-agoge*, leading outside). This mechanism which enables an individual to arrive at plausible, if uncertain, belief about a whole population, an infinite set, from actual knowledge of a few items from the set, is mysterious. The nineteenth century scientist William Whewell captures the wonder of it all:

> Induction moves upward, and deduction downwards, on the same stair.... Deduction descends steadily and methodically, step by step: Induction mounts by a leap which is out of the reach of method. She bounds to the top of the stairs at once ... (1858, p. 114)

Here deduction is portrayed in terms of descent, just as the argument or syllogism is presented on the written page - methodical, steady, safe, descending. By contrast induction is daring, creative, ascending. Incidentally, Whewell portrays *both* as female.

The philosopher Nicholas Rescher (1980) presents induction as a a tool for use by finite intelligences, a solution to the problem of providing answers to questions on the basis of limited evidence. Consider the question:

[Q] "Is it the case that every F is also a G?".

Mathematical examples include: Is the angle subtended at the circumference of a circle by a diameter always a right angle? Is n^2+n+41 prime for all $n \varepsilon N$? Are all cyclic groups abelian? Q admits four possible responses:

1. Yes, all of them are.
2. No, never - none of them is.
3. No, not always - not all of them are.
4. Don't know, can't say.

We then observe a finite set of Fs. Suppose that each observed F is indeed a G. Which of the four responses shall we make? In the circumstances, Alternative 2 is clearly ruled out. Alternative 4 merely evades the question; it adds nothing to either knowledge or belief. To choose Alternative 3 is to opt for a proposition for which we have no evidence at all. This must be worse than Alternative 1, since in this case we (merely) lack sufficient evidence. Alternative 1, an inductive inference, therefore presents itself as the preferred answer, the *optimal* or best available solution. Induction is seen to offer a responsible form of cognitive 'gap-filling'. Alternative 1 qualifies as the best *estimate to the truth* which we are able to make, on the basis of the evidence available. The plausible inference of the conclusion is *enthymematic* (information extending); missing premises are tacitly supplied in order to enable us to cross the 'epistemic gap' which separates the data from the 'answer' (to essentially infinitary questions). This is not to say that the inductive conclusion has the status of certain knowledge. Nor is it simply an uninformed guess. It is a *conjecture*. As Kneebone puts it,

> The essential difference between induction and deduction is, very roughly, that that in a deductive inference the conclusion asserts less than its premisses, whereas in an inductive inference it asserts more.(1963, p. 366)

INDUCTION AND THE MIND

Skemp (1979) gives an account of generalisation in terms of 'reflective extrapolation', a schematic restructuring akin to Piaget's notion of accommodation, which Skemp prefers to call 'expansion'. It is certainly a feature of inductive reasoning that the truth of an infinite (enthymematic) set of untested propositions is claimed, in order to expand and bind together a finite (usually small) set of items of data.

The essential finiteness of the data-base may be obscured by the manner in which it is obtained and presented. I am thinking here of dynamic geometry software such as *Cabri Géomètre*. Suppose, for example, that I create a triangle and construct its three medians. I observe that the medians are concurrent. I vary the triangle by dragging its vertices on the screen. The concurrence of the medians is a feature of every frame in the cinematographic presentation. The inductive generalisation is readily made, and with conviction. The data set appears to be continuous, uncountable. This is an illusion, since the hardware design - pixels and the like - only permits a finite, though vast, set of configurations to be calculated and displayed.

How much data *do* we need before a generalisation can appropriately be made?

> Why is a single instance, in some cases, sufficient for a complete induction, while in others myriads of concurring instances, without a single exception known or presumed, go such a very little way towards

establishing a universal proposition? (Mill, 1873, p.314).

The solution proposed by Holland *et al* (1986) is framed in terms of two individual constructs: a *default hierarchy* of related concepts, and the supposed *variability* of objects with respect to others in the default hierarchy. For example, imagine that I visit a town (call it T) for the first time. On arrival I see a red bus. I am likely to suppose (a conjecture, provisional generalisation) that, in T, (all) buses are red. Suppose, however, that I see and a tall woman or man. I am not likely to suppose that all adult inhabitants of T are tall. Why the difference? In the first case, my default hierarchy recognises 'buses in T' as a subcategory of the class of buses, and 'red' as a member of the category of colours. The observation of one red bus activates the possibility that all T-buses are red. I then make a judgement about the extent of *variability* of the superordinate category 'bus' with respect to the superordinate property 'colour', by reference to my knowledge of appropriate subordinates - buses in various towns and cities. In this case, my calculation suggests only modest variability in the bus-colour relation, and so I attach some confidence to the conjecture that all T-buses are red. In contrast, the difficulty with the tall man is now evident: body-height (from short to tall) varies considerably within adult inhabitants of any given town, and I am therefore unwilling to make any conjecture about T-adults from such a modest base of evidence. Indeed, it is unlikely that the observation of the single example (the tall adult) would even trigger the suspicion, the generalised conjecture.

The default-hierarchy/variability account accommodates the following situation from mathematics: I have just (this is fact) drawn a single pentagram, measured the (internal) angles at the vertices, and found that their sum is 180°. I am now confident that the sum of the angles of *every* pentagram, whether regular or irregular, is 180°. But I doubt, indeed I would *deny* that, the sum of the angles of every polygram is 180°.

HEURISTIC

Polya stresses the central place of induction as a paradigm for discovery, invention and plausible reasoning in mathematics, reminding us of its pedigree:

> Observe also (what modern writers almost forgot, but some older writers, such as Euler and Laplace, clearly perceived) that the role of inductive inference in mathematical investigation is similar to its role in physical research. Then you may notice the possibility of obtaining some information about inductive reasoning by observing and comparing examples of plausible reasoning in mathematical matters. And so the door opens to *investigating induction inductively*. (1954 p. viii).

Lakatos (1976, p.74) acknowledges Polya's achievement in the modern rehabilitation of a scientific-inductive heuristic for mathematics, but corrects what he sees as a Polya's "only weakness", a failure to distinguish between proof (mathematics) and explanation (science). In fact in mathematical activity proof may fulfil, at any time, one or more of a number of purposes. These include not only assurance of truth but, as in science, accounting for observed regularities.

PROOF BY GENERIC EXAMPLE

A mode of proof by 'generic example' has received some attention in the literature, (Pimm and Mason, 1984; Balacheff, 1988), and deserves much more

> The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number. (Pimm and Mason, 1984, p. 284)

The story (probably apocryphal, but see Polya, 1962, pp. 60-62 for one version) is told about the child C F Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Gauss added 1 to 100, 2 to 99, 3 to 88, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. In fact I would claim that Gauss's method demonstrates, by generic example, that the sum of the first 2k positive integers is k(2k+1). A person who could follow Gauss' method in the case k=50 would be unlikely to doubt the general case. It is important to emphasise that it is not simply the fact that the proposition that the sum 1+2+3+ ... + 2k = k(2k+1) has been verified as numerically true in the case k=50. It is the *manner* in which it is verified, the form of presentation of the confirmation. In effect the generic example triggers an *inductive inference*; that the *argument* holds in all cases.

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