THE EOUATION. THE WHOLE EOUATION AND NOTHING BUT THE EOUATION!

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This presentation addresses one teacher's method of teaching linear equations to a low ability group of pupils. It is a method that has them solving, with understanding, the most general forms of equations such as 3x-7=5x-29. The teacher treats the equations as 'mathematics', with no recourse to images from pseudo-real life, and he confronts the problems inherent in coming to a fuller understanding of the equals sign, by presenting the equation as a static entity rather than a dynamic instruction to perform some arithmetical action. Common methods of teaching the topic are summarised and the author claims that the notion of a "didactic cut", identified by some researchers, is simply an artefact of certain teaching approaches. Its effects are not seen in this classroom.

Much research (see Kieran, 1991, 1994) has been reported on the teaching and learning of linear equations - a seemingly notoriously difficult concept. Authors have highlighted a variety of ways of approaching the topic and illustrated a variety of significant cognitive problems and stumbling blocks to the learning of linear equations with understanding.

One major difficulty is that the encounter with linear equations is probably the first time that students' images for the meaning of the equals sign - built from numerical experiences - are challenged. Lins (1992) suggests "the arithmetical operations are a fundamental model for our understanding of algebraic operations; the elements in non-numerical algebras are in fact treated as if they were numbers of a different kind". Until this point, the fundamental image of '= ' as "indicating the result of an operation" has been sufficient to deal with all the symbolic expressions the students have encountered.

At first this understanding still holds good for algebraic expressions. Those presented in the familiar form of 2x-3=5 can be interpreted as "twice something take away three gives five" - a coherent verbalisation that is compatible with a variety of methods for solution. It is with the introduction of equations of the form 5=2x-3 that trouble sets in. Such an equation leads to a verbalisation of "five makes two times something take away three" which is seen to be as meaningless as the arithmetic statement "five makes eight take away three" . BUT since most school text books introduce algebra by linking it to arithmetic it should be no surprise that pupils should seek to transfer the associated linguistic understanding as well.

The verbalisations lack meaning because the temporal notion of action leading to result is inconsistent with the temporal verbal sequence of result preceding action - result (five) makes action (2x take away 3). Simply substituting the word "equals" for "makes" is not a solution. However they are verbalised, equations viewed as dynamic process inhibit future understanding.

There are suggestions in the literature (Filloy and Rojona, 1984, Hescovics and Lincheviski, 1991) that there exists a clear delineation between arithmetic and algebra at the point when students encounter linear equations with the unknown quantity on both sides of the equals sign. I would like to suggest that this particular difficulty - known as the "didactic cut" - is in reality, a notion imposed by the observer, with hind sight, to explain an artefact of particular methods of teaching. I actually believe it to be a perceived, rather than actual, phenomenon. It results from the method of approaching the solution of equations through appeal to arithmeticparallel-thinking, coupled with the introduction of expressions of supposedly increasing difficulty. Rather than being an inherent difficulty in the solution of linear equations, the cognitive obstacle is created by the very method which purports to provide a logical introduction to equation solution.

Basically there are two different approaches to the teaching of the solution of linear equations. One method is to relate the work to "real life" and the most popular method for doing this is through the notion of a balance[1]. This results in repeatedly changing the equation, by "doing the same to both side" until one has an equation that directly gives the answer. In other words: "ax+b=cx+d" --> "ax+b-b=cx+d-b" --> "ax-cx=cx-cx+d-b' --> "ax-cx=d-b"
--> "x{a-c)+{a-c)={d-b)+{a-c)" --> "x={d-b)+{a-c)".
Understanding, it is assumed, comes through association with reality. A frequent problem with this approach occurs over the image of subtraction as implying negative weights. How do you take -4 kg off a pair of scales?

The alternative approach is to work with the notion of inverse operations.
This results in repeatedly changing the equation, by use of the rule "change sides, change signs". In other words: "ax+b=cx+d" --> "ax=cx+d-b" --> "ax-cx=d-b" -->
"x multiplied by (a-c)=d-b" --> "x={d-b)+{a-c)" .
Here, understanding comes through the idea of "undoing" a series of operations to get back to the original value. In reality, of course, with neither of these methods do students start with the most difficult generalised form. They progress through a supposed order of difficulty from equations of the form "x+b=d" to "ax=d" to "ax+b=d" to "x-b=d" to "ax-b=d" to
"d=ax+/-b" and finally to "ax+b=cx+d".

Both these methods lead to the pupils "solving" an equation that is not the original given one, but some altered, in some sense equivalent equation. One consequence of this being that if pupils have really bought into the notion that the equations are "the same" they see no problem, when asked to check their answers, in simply substituting in the line above, rather going back to the early harder version of the equation.

1 I would challenge whether the two scale pan balance is, in fact, part of modem students' "real life" understanding. (see Pirie, forthcoming)

The teacher's approach which we are about to see relies upon the solving of an equation in the form in which it is given, without manipulating it in any way first. This teacher believes in mathematics as understandable, as well as 'do-able' by all his pupils, what ever their 'ability label'. He does not teach mathematics as merely a series of techniques to be mastered nor even as a tool for life after school. He acknowledges that there *are* some mathematical- actually almost exclusively numerical - skills that his pupils need for adulthood, but that given modem technology, these are in reality very few. He seriously considers it demeaning, to his pupils, to imply that they will not understand algebra unless it is presented in terms of weighing cats and rabbits! Instead of trying to conjure up pseudo 'real life' images for the mathematics he wishes to teach, he simply sets about sharing with his classes his own belief in the intrinsic value of mathematics itself. As one hears repeatedly in his classroom, he judges his pupils all to be worthy of the label of 'mathematician'. Not because of their prowess and achievements in the field of mathematics, but because 'being a mathematician' defines a way of working, a process involving errors and tribulations, that leads to an understanding of something the world calls "mathematics". The pupils are very happy with his language; he is not depreciating the word 'mathematics' nor is he patronising his pupils.

From this philosophical standpoint, therefore, the teacher confronts the notion of the meaning of the equals sign and the generalised form of linear equations head-on at the very beginning of his first lesson on such equations! The rest of this paper offers snippets from the video recordings of the lessons (shown at the meeting) to illustrate the claims I have been making concerning the efficacy of the teaching approach in achieving understanding in his pupils, and the illusory nature of the didactic cut.

Instead of the popular image of the equals sign being the pivot point of a balance, the teacher talks of a "fence". The idea of a balance carries with it dynamic images of moving scalepans and removing items from those pans, The symbol as "fence", however, suggests simply a static divider representing" .. the splitting up one side and another side" and it is mathematicians who have defined the symbol to carry the extra significance that the two sides must in some sense be equivalent, by saying "something like 'this side must have exactly the same number as this side'.". The teacher's message is that an equation is a single entity. It is a statement of a static fact. It is not an order to 'do something and get something'. It has no temporal existence.

In the lessons prior to the introduction of linear equations, the pupils have been working on arithmagons, and one of the noticeable features of the teacher's lessons is that he always ties them to previous work. In this case he reminds the class of the notion of the fence with

Teacher You've been dealing with that [the fence] ... you were saying [in the arithmagons]'A' plus 'B' must equal 6, let me check, four add two is six, yes, that's correct. Or you might have said five add three is eight, ah, that's not six, so it can't be right. .... The puzzles today ... [are] going to be fill in the gaps so that one side of the fence .. is equal to, contains the same amount as, the other side of the fence.".

The very first question that the teacher puts up on the board is G G 18= D +53. The crucial features of this are, firstly it is in the most general form, and secondly it has no obvious solution. It is not given in 'easy numbers'. He does not want them to be able to 'see' that answer, he wants them to think about how to get at a solution. He invites the pupils to seek a number that would go in the boxes to "make the statement correct", but first he suggests

Teacher You might want to say, just to get us warmed up, a value that doesn't work. That would be a reasonable thing to start.

After giving the pupils a minute to think about the problem and each try out a number, the teacher takes a suggested number, asking as it is given, "is that a correct statement or not?", writes it in the boxes and asks the class to check that it is indeed incorrect. After taking a couple of suggestions he comments

Teacher Maybe now, as good mathematicians, we'll be using the two answers we've got to suggest roughly what the correct answer is.

He hints that looking at the totals they have on each side of the fence leads to the idea that the answer will be a larger number than they have already tried. The industry of the pupils as they work on the problem is obvious and the excitement of two girls, Marie and Joan, as they get the correct solution, 35, is palpable. The teacher's comment then is

Teacher That one's taken us about 15 minutes, nothing wrong with that but I do want you as mathematicians to try and come up with a method which gets us there a lot quicker .... As you go through the questions I'm giving you, is there a short cut? is there a quick way? is there a mathematical way?".

At this moment Marie suggests

Marie If you just add the last two numbers, 35 and 18, it should make 53 ... so you've got 18 you add something, you try and add something with it to make the last number" .

The teacher accepts the suggestion, saying "it's good", but does not try to impose it on the rest of the class. Later in the lesson we hear Joan, working with a calculator on the problem G G27= G50.

Joan *(button punching as she talks')Right.* 27 add 24 equals 51. 27 add 23 equals 50. 27 add, yeah, 27 add 23 equals 50 so it's 23. *(to an interruption by Marie)* No, wait a minute. This might be wrong, 23 add 23 add 27 equals 73, erm what was it, 23 add 50 equals 73, yeah, it <u>is</u> right. *(calling out loudly)* Sir, we know how to do it now. It worked, Marie's thing.

Notice that she has automatically checked her solution in the original equation. She does not work with an altered version. She is thinking of the teacher's question "Is this a correct statement?"

Later in the same lesson Marie meets a minus for the first time: G 0-14= D +9. She explains to the teacher that her method still works:

Marie *(in a voice ofconfidence)1* decided how would you make 9 out of 14. So I decided you take away 5, but I was doing it wrong, you see *(with a broad grin),* cos if I needed a higher number there to take away 14 to make 9, so I made it 23.

She is explaining that the first time round she just did the wrong piece of arithmetic - which had been instantly obvious to her because she automatically checked it with the original equation.

A couple of lessons later, the class is working with equations in their symbolic form, and although most of the pupils are comfortably working with the complete general equation, the teacher offers those who wish to use it, a way of writing the solution with an intermediate step of the form ax=b. Marie and Joan, however, have been going from the equation to the answer without needing to re-write the equation in an intermediate or simpler form. The calculator gives them the power to hold onto the whole solution process and so come up with the solution of the original equation. Following the teacher's intervention, they start to back track on their thinking in order to write it down! An illustration of this comes from their working on the problem

11 +5b=3b+25.

Joan: "25 take away eleven equals 14 shared by 2 is 7. *(pulling her book towards her , she writes the question, leaves a line's gap ,and writes "b=7")* So this one's going to be 2b, how do you write that down? ... *(filling in the space she left)2b* equals 14 and so *(reading her finalline)box* would equal 7".

It is hard to remember that this is a below average ability set, particularly when one considers the following exchange between the teacher and Marie, and the extent of the understanding and facility to think algebraically that it reveals. It is the body language of Marie as she makes her final response that gives the clue to the low personal esteem in which she has been previously led to hold her mathematical abilities.

Marie *(talking to Joan)* That's a take away so it'd be at the end. It's near enough a normal one.

Teacher *(overhearing her)* When you said: that's a normal one, what did you mean by that?

Marie Is it like the beginning? *(pointing to the first equation on the sheet: 7b+3=4b+21)*

Teacher And what makes that 'normal' for you?

Joan Because it's easy. *(laughs)* the other ones are a bit complicated.

Teacher What makes it easy?

Marie The number that side *(points to LHS)* is higher than the number that side *(points to RHS),* the boxes, and there's no take always in it.

Teacher Does that make it easier for you?

Marie *(lowering her eyes to her desk, nods)*

Teacher Why?

Marie *(hanging her head and almost inaudible)* 'cos I can add better than take away.

It is the arithmetic, not the solution of the equation that she sees as hard. The irony of her last remark is, of course. that if the equation has '+' then what she actually has to do - and has successfully been doing - is 'take away', when the equation has '_| she herself has to add! The tragedy is that in spite of her notable success in this teacher's class, she still conjures up negative images of her ability when faced with the subtraction sign.

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