TEACHERS' USE OF GENERIC EXAMPLES Liz

Bills

Centre for Mathematics Education, Open University

*I describe my understanding of a teaching technique which I have labelled as
"generic example" and the responses made to this description by a group of
teachers. I consider some similarities and differences between my examples of
the use of the technique and the teachers 'examples.*

I am going to define a transformation T as follows:

$$T:\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

I want to look at the effect of this transformation on a particular set of points, namely the line $y = 2x + 1$. I'm interested in whether the image set is a particular curve - that is whether there is an algebraic relationship between the x and y co-ordinates of any point in the image set. If I call these x and y co-ordinates X and Y, then:

$$X = 2x + 3y + 4$$
$$Y = x + 2y + 5$$

But if the point (x, y) lies on the line $y = 2x + 1$ then

$$X = 2x + 3(2x + 1) + 4 = 8x + 7$$
$$Y = x + 2(2x + 1) + 5 = 5x + 7$$

So
$$Y = \tfrac{5}{8}(8x + 7) - \tfrac{5}{8} \times 7 + 7$$
$$= \tfrac{5}{8}(8x + 7) + \tfrac{21}{8}$$
$$= \tfrac{5}{8}X + \tfrac{21}{8}$$

I have found a linear relationship between X and Y and so I know that the image of the line $y = 2x + 1$ under the transformation T is another straight line.

Having presented this narrative, I might ask my students to "Prove that affine transformations map straight lines to straight lines". How might hearing the earlier narrative be a help to them?

One story I might have is that my narrative portrays a procedure for establishing the nature of the algebraic connection between the co-ordinates of an image point under a transformation. My narrative treats a particular transformation acting on a particular set of points. I am asking my students to see the particular transformation as representative of a class of transformations and the particular set of points as representative of a class of sets of points, that is straight lines.

I want to explore this issue in two ways. First I will describe what happened when I asked a group of teachers to give me their examples of using a teaching technique "similar" to the one I have outlined. Secondly I will discuss some distinctions between the examples they gave me and my own examples.

A few months ago I presented the example with which I started this writing to a group of teachers and then asked them to produce a proof that affine transformations map straight lines to straight lines.

I followed it with a description of the way in which I had taught a class about finding the coordinates of a point which divides a given line segment in a given ratio. I said that I had chosen a particular line segment, from (1, 2) to (5, 7) and a particular ratio, 3:5. I had used a question and answer sequence to demonstrate a procedure for obtaining the co-ordinates of the new point. I stressed that I had used the particular example to talk about the general case. For example, I had referred to "the x-co-ordinate of the first point" rather than **1,** and I had not completed any calculations but left them in the form 1 **+** 3~5(5 - 1). I left it open to the class whether they

went on to derive a formula or simply to remember the method. I used the phrase "generic example" to refer to the technique I was aiming to exemplify. This term was borrowed from Mason and Pimm (1984) although I am using it with a different shade of meaning.

With the teachers I also discussed an occasion on which I had worked with a student who was expressing difficulties in working on this question: "Find the equation of the line with gradient *M* which passes through the point *(p, q)".* I had tried to help him by asking him first to find the equation of a line which had gradient 4 and went through the point (2, 3).

I asked the teachers to consider whether they used any teaching approaches which were similar to the three I had demonstrated and described so that we could share experiences the next time we met.

At our next meeting the following descriptions were given by five of the teachers:

- in teaching an upper sixth statistics class how to use tables to calculate probabilities associated with the Normal distribution, one teacher used particular numbers rather than statements of the form $P(z < -a) = 1 - P(z < a)$.

- in working with a lower sixth class on solution of simple trigonometric equations, for example 2sinx = --.*j3,* another teacher encouraged his students to solve the equation in degrees, with which they felt comfortable, and convert the solutions into radians afterwards if necessary .

- a third teacher "covered the blackboard with little examples" in order to encourage her lower sixth class to express the rule for differentiating a power function. She wrote, for example, $x^2 \sim 2x$, $x^6 \sim 6x^5$.

- a fourth teacher said that she had previously used precisely the method I described for teaching about dividing a line in a given ratio.

- in a year ten class a fifth teacher suggested that students who could not see how to manipulate the algebraic expression to solve, for example, $12 = \frac{\sim}{x}$, should substitute numbers which make it easy to see the answer, In this case perhaps $5 = 11$. Then, seeing that the answer x could be obtained as $15 + 5$, they might apply the same structure to the original equation to get $x = 2 + 12$.

Some of these descriptions did not fit with what I meant by "similar teaching approaches", but rather than discard them I want to use them as a means of exploring what the teachers perceived as the relevant features of my examples.

First, one primary aim in each of my examples was to assist students in expressing the general case algebraically. **In** the case of the affine transformation I was asking for a general proof in the expectation that it would involve algebraic expression of a general affine transformation and a general straight line. In case of the division of the line segment my main aim was to enable students to solve similar problems in the future but a subsidiary aim was that they might express the method for calculating the co-ordinates of the point as an algebraic formula. In the third case the algebraic expression of generality was an overt aim.

In all of the teachers' examples however, solving similar problems in the future was a paramount aim, and only in the differentiation example was an algebraic formulation explicitly sought.

Secondly, in my examples the eventual general task was first made more particular, whereas in the case of the trigonometric equations and the year ten equations the task was first made not more particular, but simpler or more familiar.

Thirdly, my aim in each of the three examples was that students should understand the reasoning behind the procedure used in the particular case and be able to transfer that reasoning to the general case. **In** contrast, the differentiation case asked students to see the pattern in the particular examples presented to them but not to know how these particular results arose. Another way of expressing this difference is to say that my examples aimed to appeal to students' deductive reasoning whereas the differentiation case appealed to students' inductive reasoning.

I want to summarise these distinctions by saying that whereas my emphasis was on helping students to move from particular to general, and to demonstrate that they have done so by expressing the general case, the teachers' emphasis was on enabling the students to operate competently with the particular. This difference in emphases does not necessarily correspond to a difference in the way students perceived the tasks. **In** particular, whereas my aim was for students to understand the structure of the solution procedure in the particular case and to transfer the structure to the general case, it may be that students would use the particular case as a template, merely substituting the general expressions for the particular without assimilating the method.

In looking for similarities between my examples and the teachers' examples, I fmd that we are all expecting students to see an essential "sameness" between examples. In each of my three examples I am expecting students to see that the general case is "essentially the same" as the particular case that I have worked through. In its most extreme form this awareness allows students to see, for example, that they don't need to work through the proof that all affine transformations map straight lines to straight lines because the working would be "essentially the same" as that which I had already done. This appreciation of sameness is an expert awareness which legitimates the use of "similarly" and "by symmetry arguments" in proofs. It could also rest happily among Arcavi's list of the constituents of "symbol sense" (Arcavi 1994).

In the teachers' examples there is similarly a need for an appreciation of sameness on the part of the students. The approach to teaching the normal distribution relies on students seeing the sameness of the method for finding $P(z < -3)$ and $P(z < -2)$. The use of degrees instead of radians relies on students seeing that the same solution method works for both. The technique for eliciting the method for differentiating a power function relies on students seeing that all the examples are of "the same thing". Finally the technique used with year ten students working on equations relies on an appreciation that $12 = \sim$ and $5 = 11$ are the same type of equation, in the x x sense that what is permissible for one is permissible for the other and that a solution method for one will work for the other. Below is an account of an incident which illustrated for me that this kind of appreciation may be taken for granted by teachers.

> Martin does an example of adding algebraic fractions on the board. The task is to simplify $x + \frac{1}{x}$. This is rewritten as $\frac{1}{x} + \frac{x}{1}$. and then a common denominator of x chosen, giving $\frac{x^2}{x} + \frac{1}{x}$. At this point Joe wants to cancel the x with the x^2 to leave $x+1$. Martin tries to explain why this cannot be done by giving the example of $\frac{4+12}{8}$, but Joe rejects this example because 4 is not 8 squared.

Here Martin's example is not an example for Joe. He is more concerned with the differences between the original example and Martin's example than by their sameness.

In summary, my concern with the move from the particular to the general was not seen as the most important feature of the "generic example" by the teachers. However the requirement for students to appreciate the sameness of two or more examples was a feature common to the situations we described as "generic examples".

Bibliography

Arcavi A. 1994 "Symbol Sense: Informal Sense-Making in Formal Mathematics." For the Learning of Mathematics 14.3 : pp24-35.

Mason J, Pimm D. 1984 "Generic Examples: Seeing the General in the Particular." Educational Studies in Mathematics 15 : pp277-289.