JUSTIFYING AND PROVING IN THE MATHEMATICS CLASSROOM

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In the current post-Cockcroft era the learner of school mathematics engages with proof in an informal way. Through personal investigations and problem solving the learner is introduced to process skills such as conjecturing, generalising, and justifying. Recent research (Cae and Ruthven, 1994) *reports that the proof-practices of advanced learners are predominantly empirical. These practices are nevertheless valuable: such proofs convince their [roducers and, under certain circumstances, empirical proof-practices may be precursors to deductive ones.*

Background: Mathematics is not just about identifying what is tru or what works but also about proof. Unlike other kinds of proof, mathemaical rpoof is a dialectic “which follows the reaching of conviction" (Bell, 1976). For the learner of school mathematics the meaning of proof first comes from everyday life and/or from experiments in science where proof is about *reaching* rather than *following* conviction and where conviction may come from intuition or by weight of evidence. Thus the early proof practices of learners are empirical (Van Dormolen, 1977; Balacheff, 1988). Such 'proofs' are legitimate in the eyes of these learners because " ... they are recognized as such by their producers" (Balacheff, 1988). But if these proofs are to have public meaning then they have to convince other learners, especially the doubters; that is, the empirical proof practices ought to develop into deductive, logical ones. Some recent research (Coe and Ruthven, 1994) suggests that this may not always happen. The conjecture that my recent pilot study attempted to test was: The transformation of empirical proof-practices into deductive ones is dependent on an appropriate learning experience and a careful nurturing of prototypical proof-practices.

The pilot study The school chosen for the pilot study was a large suburban secondary school. The school's mathematics department has for many years enthusiastically used and contributed to the *SMILE* individualised scheme for learning mathematics whose curriculum approach places much emphasis on guided discovery, problem solving, and investigations. A major proportion of the learners tasks are about establishing or identifying generalities and the potential for explaining or justifying them is frequently present even if the requirement to do so is not. In 7 separate visits to the school I talked to 22 learners from different classes but mainly from year 10. These were chosen by their class teachers to represent a range of abilities. I tape-recorded one-to-one conversations with learners about their attitudes to discovery and investigation type mathematics and their reasons for believing the rules and formulae they discovered.

The 22 learners that I conversed with displayed a range of proof attitudes and practices in their work. Using descriptions by Balacheff (1988), their proof practices ranged from *naive empiricism* (proof by weight of evidence) to proof by *crucial experiment* (checking the conjecture against a typical case) to proof by *generic example* (using mathematical properties and structure in a generic case to explain why the conjecture is always true). Though it must be said that the incidence of the latter was limited and evident in the work of just two learners. In the following I describe relevant features of learners' proof attitudes and practices by means of selected portions of transcripts.

The following represents part of a conversation I had with Alex, a year 9 learner, who appears to belong to the category of learners who "only provisionally accept something for which the evidence is purely empirical" (porteous, 1990). I talked to Alex about the result 'the sum of two odd numbers is an even number' that he had already discovered in his classwork and asked him explain why he believes that the result is always true.

> Alex - Because there's some cards in class and it's to do with odd numbers and evens.
>
> Me - OK. But how can you explain to someone else that you will always get an even number?
>
> Alex - We can't really .... there's millions of ways using loads of numbers. Sometimes it might not be what you expect it to be ....you might get really big numbers and come up with odd numbers instead.

Alex's initial response is to invoke the knowledge gained in doing a previous task - card 240: Odds and Evens Tables. There, either by intuition or by an empirical method, he determined that the sum of two odd numbers is an even number. Within the context of his everyday mathematics experience which deals with numbers that are not 'big' he is reasonably sure that the assertion is true. However, when asked to explain why it is *always* true Alex becomes unsure, now becoming aware of the wider mathematical world where 'big' numbers exist. In that world Alex recognises that the assertion may not be true - he cannot yet be absolutely sure that the sum of two odd numbers is an even number. Not all learners exhibited this level of awareness when responding to this question. Seven learners accepted that the 'odds and even' rule was true either because of the weight of empirical evidence or because it was obvious. Bhavesh, a year 10 learner, displayed both sets of thinking.

> Bhavesh - Well, I'm not going to explain it, it just is, and (..........).
>
> Me - It is?
>
> Bhavesh - I don't know how to explain it, the only thing is to give evidence to prove it, that should (.......) speak for itself.

Bhavesh appears to believe that the assertion is true on intuitive grounds - "it just is" - in much the same way a young child might believe that the shortest distance between two points is a straight line. rt could be argued that Bhavesh does not know how to explain why the assertion is true because it is difficult to explain the intuitively obvious 'fact'. With the 15 other learners, however, the invitation to explain the 'odds and evens' rule invoked positive responses and these ranged from a recognition of the need for a non-empirical explanation to convince doubters (as with Alex) to acceptable mathematical proof As an example of the latter Carla, a year 9 leamer, said the following:

> Carla - ..... Ifyou take 1 away from that odd number it will be even, so if you add the 2 numbers left over together, that makes an even number and three evens make an even number.

Such proofs were not always evident in learners discovery or investigation type tasks. In general learners accepted their self discovered rules and formulae on the basis of the *crucial experiments* that they performed. This is illustrated by the following conversation with Dee, a year 10 learner, about the circumcircle of a triangle.

> Dee - There's a triangle and you have to do the perpendicular bisector of every single line of the triangle and draw (..........) all the lines meet in the middle. So then if you get a compass and bring the pencil into the point of the triangle that drawing is that circle around the whole triangle.
>
> Me - Oh, that's fascinating. So not only do these perpendicular bisectors all meet at a point, in the middle of the triangle, but that point is the centre of the circle which touches all comers of the triangle. (Dee - Yes). So, you've done it once, and done it twice?
>
> Dee - I took any triangle.
>
> Me - I see, 1,2,3,4,5,6, 7, 8 triangles. Eight pieces of evidence.
>
> Dee - The card said try it out so I just kept on doing it until I was convinced.
>
> Me - In these types of card how many examples on average does it takes to convince you personally?
>
> Dee - About 5 ....not what's in the cards.

Dee is careful in choosing typical triangles - "I took *any* triangle" - for the proof by *crucial experiment.* Dee says that, on average, she needs to perform 5 *crucial experiments* of her own to convince her of the truth of assertions. This number varied from individual to individual: The largest number that learners said they used in their work was 10 *crucial experiments* of their own. Most rejected 3 *crucial experiments* as being inadequate. The number varied from problem to problem: "It depends on what type of thing you're trying to prove" was one response.

Conclusions Despite the relative absence of non-empirical proof practices most learners were aware of the explanatory function of proving or justifying. For some learners this awareness appears to have arisen by an internal critical process. For others it appears to have arisen via the appearance of the words "Justify: can you convince someone that you've solved the problem and understand why?" in the flow diagram for investigations that adorn classrooms and/or via the knowledge that such a practice would merit a higher level: "The teacher's told us we'll get a higher level if we explain why the rule works." Furthermore, given the invitation to explain a learner *accessible* conjecture, they seemed willing to volunteer an explanation or consider the invitation at the very least. For example, the majority of learners were willing to explain why the sum of two odd numbers is always an even number even if some of the explanations may have been clumsy or involved faulty reasoning (such as, 'an odd number is a minus and an even number a plus and two minuses gives a plus'). It appears that there is a *potential* for development of these learners' proof practices. Frequent explicit and learner *accessible* invitations to prove in the course of their normal mathematics work may help realise this potential. Further research in this area of proof *potential* is being considered.

References

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