Abstraction, Justification and Generalisation: How high school students tackle open-ended exploratory mathematical problems

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In this paper, I examine how three students from India, aged 16 to 18, reason through the open-ended art gallery mathematical problem. This study looked at how students created abstractions through vertical reorganisation. Generalisations made by students were classified as empirical and theoretical, and the generalisations based on schema were classified as expansive, reconstructive and disjunctive. Students initially made a disjunctive generalisation, that only one guard would be needed, followed by reconstructive generalisations as they tried more examples of different shapes of art galleries. Vertical reorganisation was also observed as students created and worked on new conjectures. The students primarily used theoretical generalisations and were able to justify their claims. While working on the problem, participants created conjectures related to blind spots and the upper limit of the number of guards required.

Keywords: abstraction; generalisation; vertical reorganisation; open problems.

Introduction

Generalisation is a crucial aspect of mathematics (Zazkis et al., 2008). Here, generalisation refers to the process of extending an argument which is observed in a few cases to a larger group (Harell and Tall, 1991). The importance given to generalisation is not limited to higher mathematics; according to Davydov (1972/1999), "developing children's generalisations is regarded as one of the principal purposes of school instruction" (p.10). A closely linked process is abstraction, which can be seen as the process of creating new mathematical structures (Hazzan & Zazkis, 2005). The National Curricular Framework (NCF, 2005) in India states that mathematics education should "develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstraction" (p. 42). The emphasis on abstraction has been retained in the current National Curricular Framework (NCF, 2023). Ramanujan and Subramaniam (2012, p.10) observe that there is very little research in these areas in India. Although this observation was made over a decade ago, I still found very little research addressing these processes in the Indian context.

Abstraction and generalisation are closely linked. While some authors would consider these processes as identical, in this study, I adopt the distinction made by Mitchelmore and White (2007). According to these authors, generalisation is seen as an important component of abstraction; multiple generalisations are synthesised together during the process of abstraction. In this study I am interested in how students from India create abstractions and generalisations.

Literature review

In this study, I use the theory of abstraction proposed by Davydov. Abstraction is understood as moving from underdeveloped to more developed ideas (Hershkowitz et al., 2001). As students move to more developed ideas, vertical reorganisation is important; this involves using previously encountered meanings and ideas in mathematics to create a new understanding (Dreyfus, 2015). Therefore, while students initially may have inconsistent, incomplete or contradictory ideas, as students' ideas develop, they will have a more complete understanding.

To understand the generalisation created by students, I adopt the categorisations of generalisation proposed by White and Mitchelmore (1999) and Harel and Tall (1991). According to White and Mitchelmore (1999), a distinction can be drawn between two different methods of generalisation: empirical and theoretical. Empirical generalisation is based on patterns observed over multiple examples and is particularly useful for producing conjectures. Here, conjectures are formed based on common features or qualities of the examples (Zazkis et al., 2008). In contrast, theoretical generalisation is based on the underlying structure, procedure, or meaning given to the examples. In this form of generalisation, extracting and abstracting invariant properties of the examples are important (Zazkis et al., 2008). Therefore, while empirical generalisations are powerful in forming conjectures, theoretical generalisations are important in understanding and justifying those conjectures.

In contrast, Harel and Tall (1991) categorise generalisations as expansive, reconstructive, and disjunctive. Expansive generalisation would involve using an existing schema to extend it to a broader set of situations. On the other hand, reconstructive generalisation would occur when existing schemas are modified so that they can be applied to larger groups. Finally, disjunctive generalisations occur when individuals encounter new situations and create schemas which did not previously exist.

I think that these two categorisations proposed address different aspects of generalisation, White and Michelmore focus on the reasoning behind the generalisations, while Harel and Tall (1991) examine how pre-existing schemas are used in new situations. I, therefore, classify students' generalisations using both frameworks, putting together these theories to form six categories of generalisation (see Table 1). In this study, I aim to understand how secondary school students' approach and work on mathematical problems. My research question is: What aspects of abstraction and generalisation can be observed when secondary school students work on a new mathematical problem? I analyse students' responses to understand how abstraction occurs through vertical reorganisation and categorised the generalisations made by students.

Table 1. Six categories of generalisation

Expansive empirical generalisation	Expansive theoretical generalisation
Reconstructive empirical generalisation	Reconstructive theoretical generalisation
Disjunctive empirical generalisation	Disjunctive theoretical generalisation

Methods

The research design was a small-scale qualitative inquiry and involved interviews with three students aged 16 to 18. All the participants attended an alternate school in Bangalore, India, and had completed the IGCSE mathematics examination. At the time of the study, participants were doing their A-levels in mathematics or related subjects, such as physics. The interviews were semi-structured and lasted approximately half an hour. During each interview, students were given the following question in writing

> Supposing there is an art gallery where each wall is a straight edge. What is the minimum number of guards we would need so that the guards can observe the whole gallery together? (Here, every guard can observe anything in their view). Try it out for different shapes of galleries, are there any patterns that you notice? Will these patterns always be true? Can you explain?

The problem presented to the students is based on the Art Gallery Problem, originally proposed by Victor Kleen in 1970 (Petruzelli, 2022) and originally solved by Chvatal. This problem is a low-threshold, high-ceiling problem, allowing all participants, regardless of their mathematical ability, to engage with it. By selecting an open-ended problem, one which could be approached and solved in multiple ways, I intended to explore how higher-order thinking was used by students (Edson, 2017) and how examples supported their thinking.

A clinical interview approach was used to understand students' thought processes (Ginsburg, 1981). Specifically, I was interested in how students generalised and how abstraction occurred. To gain deeper insights into participants' reasoning, generalisations, and justification, probing questions were used (Ginsburg, 1981). For each of the patterns students identified, they were asked if the patterns would always hold, and to explain their thinking. Additionally, the examples students used were recorded to understand their approach to the problem. Each interview was recorded and transcribed to produce an intelligible verbatim transcript, where the emphasis was placed on capturing the exact meaning and essence for analysis (McMullin, 2023). As previously outlined, generalisations were categorised using the frameworks of White and Michelmore (1991) and Harel and Tall (1991). The types of generalisations were categorised as empirical or theoretical, and the use of pre-existing schemas while creating generalisations was classified as expansive, reconstructive and disjunctive. Instances of abstraction, through vertical reorganisation, were also identified and analysed. In the following section, I present and discuss some key generalisations and conjectures developed by the students during the interviews.

Findings

Initial approach

Each of the participants initially looked at the problem using convex shapes, beginning with a square and then making general claims about other 'similar' shapes. Participants observed that only one guard would be needed for triangles, squares, rectangles and hexagons, which are all convex. Here, however, participants did not explicitly mention that the shapes were convex; this is interpreted based on the nature of the shapes they drew while making this conjecture. This appeared to be an instance of disjunctive theoretical generalisation for all participants, as they constructed a new schema in response to a problem they had not previously encountered. When asked follow-up questions, all the participants realised that their initial conjecture, that only one guard would be sufficient, did not hold for more complicated shapes. Through this process, participants appeared to shift towards reconstructive theoretical generalisations, recognising that the original conjecture they created held only for a limited set of shapes rather than all shapes.

This movement from encountering a new problem, to making an initial conjecture about the number of guards needed and then recognising the limitations of the conjecture, appears to be vertical reorganisation as participants create abstractions. When participants realised the conjecture did not always hold, and made further conjectures, participants seemed to be vertically reorganising and developing their understanding. During this process, participants moved from an underdeveloped understanding to a more developed one. While constructing more complicated shapes, all participants had the idea of introducing blind spots to increase the number of guards required. However, the strategies they used to create these blind spots varied.

Blind spots

As participants continued working, all three participants developed the concept of 'blind spots', areas within a shape that would be hidden from a guard's line of sight. They applied this idea to construct shapes that required more than one guard and later used it to try and maximise the number of guards needed for a shape with a fixed number of sides. To create blind spots, participants used various strategies, such as designing corridors that extended in opposite directions and incorporating very narrow sections into a shape.

When prompted to explore more complex shapes, participants used three strategies to construct these shapes. The strategies used were adding corridors to an existing shape (see fig. 1a.), removing sections from an original shape, either a part from a side (fig 1b.) or from the middle (Fig 1c.), and adding internal walls to a simpler shape (fig. 1d.).

a. b. c. d.

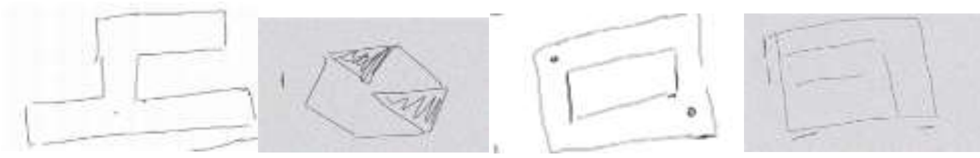

Figure 1: Shapes participants created.

Participant S3 also used the idea of blind spots to argue that it was not possible to construct a six-sided shape that needed three guards.

> S: OK, so six sided which needs 3, right?
>
> I: Yes.
>
> S: Ok, I don't think it really works, because you need three corners and two lines attaching the three corners, so 6 lines, and then you have to get them (the sides) to attach, so it (the shape) has like 3 pointed corners, so the guard can't see.
>
> I: Sorry, can you explain that again, I didn't understand.
>
> S: So, you would need…Something like 3 pointed corners like that (See fig. 2) so the guard can't really see into it, like one corner, but then, that means, that's already 6 lines, but they (the sides) all should be attached so…You can't really find a six-sided shape with three guards.

Figure 2. Trying to create a six-sided shape that would need 3 guards

In this excerpt, the participant used the idea that creating a blind spot required at least two sides. Following this part, I asked S3 why an additional side would be necessary; she explained that if the blind spots were simply connected to one another without additional sides (see fig. 2), they would no longer remain as blind spots. In this instant therefore S3 conjectured that the upper bound for the number of guards for a six-sided shape would have to be two.

Generalisations

The generalisations made by participants were theoretical, and participants were able to justify them. Below is an example of one such generalisation.

> I: Is there a maximum number (of guards) you'll need for an eight-sided shape?
>
> S: Hmm, there could be actually, I think, I'm sure if you try out … max would be 8 (guards), right? One for each side, like if you were to have all of them (the sides) like this lined up (fig. 3), you need 8, depending on how many ever… I think this is the most extreme situation in some sense. I don't think you could need more than eight guards because this is the most cut up it (the gallery) could get.

Figure 3. The most 'extreme situation' with eight sides.

Here, S2 used their earlier ideas to justify a general claim about the maximum number of guards that might be needed. The participant was able to explain their reasoning, which was based on the underlying structure of the shape rather than merely a pattern, making it a theoretical generalisation. Here, the student seemed to be creating the most 'extreme situation'. Following this, I asked the student probing questions about the structure of the gallery, such as: What if all the walls had to be connected? What if a boundary had to enclose all the walls? The participant responded to this by saying that this was the most extreme situation, and therefore, in the following cases, fewer guards would be needed.

Across the interviews, the generalisations made by participants were theoretical in nature, and they were able to explain the reasoning behind their thinking. No instances of empirical generalisation were observed, as participants appeared to focus less on identifying patterns and more on the underlying structure and reasoning that gave rise to the patterns.

Conclusion

This small-scale qualitative inquiry studied how three high school students approached and worked on the art gallery problem, an open-ended, exploratory mathematical problem. Particular attention was given to the generalisations and abstractions formed by students. All three students initially conjectured that one guard would be sufficient for all shapes, but they subsequently recognised that this did not hold. While each student used the idea of creating blind spots to challenge their initial conjecture, their approaches to creating the blind spots were different. Two particularly interesting conjectures were created by the students. One student proposed an argument for why a six-sided shape could never require more than two guards, while another student constructed an upper bound for the number of guards required for any shape based on creating the most 'extreme' shape of an art gallery. Across all participants, the generalisations were primarily theoretical, with students being able to justify and explain the generalisations that they created. This study examined how students from one school approached a single open-ended exploratory question. Future research could explore other such questions to investigate whether abstraction and generalisations vary, and how students with different mathematical experience approach problems.

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